SHELLABILITY AND REGULARITY OF CHAIN COMPLEXES OVER A PRINCIPAL RING

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ABSTRACT. The goal of this paper is to generalize some of the existing toolkit of combinatorial algebraic topology in order to study the homology of abstract chain complexes. We define *shellability of chain complexes* in a similar way as for cell complexes and introduce the notion of *regular* chain complexes.

In the case of chain complexes coming from simplicial complexes we recover the classical notions but, in contrast to the topological case, in the abstract setting shellings turn out to be a weaker homological invariant. In particular, we study special chain complexes which are *cones*, and a class of regular chain complexes, which we call *totally regular*, for which we can obtain complete homological information.

1. INTRODUCTION

This paper aims to generalize the notion of shellability to abstract chain complexes. A simplicial complex is shellable if there is an order of its maximal simplices $F_1, F_2, \ldots, F_t$ such that $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is a pure simplicial complex of dimension $\dim F_k - 1$ for each $1 \leq k \leq t$ (cf. Kozlov, 2008, page 211). Now, any simplicial complex gives rise to a chain complex (cf. Hatcher, 2008, page 104), therefore one is naturally led to ask for an algebraic counterpart of shellability. As far as we know, this question has never been studied before. As a point of interest, it turns out that the algebraic situation is more complicated than the combinatorial case. In the end, so called totally regular chain complexes (a special kind of regular ones) turn out to be an analogue to shellable simplicial complexes as they have the same homology.

In Section 2 we shortly present some basic facts on principal rings and free modules before we introduce abstract chain complexes in Section 3. Thereby, we define critical basis elements as an analogue to spanning simplices in simplicial complexes (cf. Kozlov, 2008, page 212) and compute the homology of pure chain complexes having critical basis elements. Section 4 contains a small excursion about acyclic chain complexes and cones.

Both the last sections deal with our main topic: shellable and regular chain complexes. After defining shellability for chain complexes in Section 5, we prove the existence of a special shelling which we call monotonically descending. We also show that i-skeletons of shellable chain complexes are shellable themselves. In contrast to shellable simplicial complexes, the homology of shellable chain complexes is not known in general. Therefore, we introduce regular chain complexes as a special class of shellable chain complexes in Section 6. As above, any i-skeleton of a regular chain complex is also regular itself. Finally, we compute the homology of special regular chain complexes which are called totally regular.

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2. Preliminaries: Principal Rings and Free Modules

A *principal ring* is a commutative ring with 1 which is an integral domain whose ideals are all principal (cf. Bosch, 2004, page 35). If M is a free module over a principal ring R, then every submodule of M is free (cf. Lang, 2002, page 146).

We will mostly use finitely generated modules, i. e. modules of the form $M = \bigoplus_{i=1}^{n} Re_i$. For these modules we define:

- The *generating number* $\operatorname{gen}_R M$ is the minimal number of elements which generate M (cf. Oeljeklaus and Remmert, 1974, page 90):
$$\operatorname{gen}_R M := \min\{n \in \mathbb{N} \mid M \text{ is generated by } n \text{ elements.}\}$$
- The *degree of freedom* $\operatorname{dgf}_R M$ is the maximal number of linearly independent elements in M (cf. Oeljeklaus and Remmert, 1974, page 110):
$$\operatorname{dgf}_R M := \max\{n \in \mathbb{N} \mid \text{There are } n \text{ independent elements in } M.\}$$

The next two theorems are quite helpful and we will use them later in Section 3.2 (cf. Oeljeklaus and Remmert, 1974, pages 112 and 115).

Theorem 2.1. *Let R be an integral domain. Let M and N be finitely generated R-modules. For any R-linear mapping $\varphi\colon M \to N$ holds:*
$$\operatorname{dgf}_R(M) = \operatorname{dgf}_R(\ker\varphi) + \operatorname{dgf}_R(\operatorname{im}\varphi).$$

Theorem 2.2. *Let R be an integral domain. If M is a finitely generated free R-module, then* $\operatorname{gen}_R M = \operatorname{dgf}_R M$.

3. About Chain Complexes

In the following let R always be a principal ring. We want to introduce now the basic concepts of chain modules over a principal ring before we define an analogue to a *spanning simplex* (cf. Kozlov, 2008, page 212).

3.1. First Concepts.

Definition 3.1. For every $\nu \in \mathbb{N}$ let C_ν be a free module over R with basis $\Omega_\nu := \{e^\nu_1, \dots, e^\nu_{k_\nu}\}$, $k_\nu \geq 1$, or $\Omega_\nu := \emptyset$. Let
$$\begin{aligned} &\partial_\nu\colon C_\nu \to C_{\nu-1}, \quad \nu \geq 1,\\ &\partial_0\colon C_0 \to 0 \end{aligned}$$
be R-linear mappings so that $\partial_\nu \circ \partial_{\nu+1} = 0$ (i. e. $\operatorname{im}\partial_{\nu+1} \subset \ker\partial_\nu$). Then we call
$$\ldots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$
a *chain complex* (C, Ω) with basis $\Omega := \bigcup_{\nu\in\mathbb{N}} \Omega_\nu$. The free modules C_ν are called *chain modules*, and the mappings ∂_ν are *boundary mappings*.

A chain complex (C, Ω) is *finite of order d* if $C_\nu = 0$ for all $\nu > d$ and $C_d \neq 0$.

Remark 3.2. We always consider chain complexes C together with a fixed basis $\Omega = \bigcup_{\nu\in\mathbb{N}} \Omega_\nu$ and we do not want to change the basis of any chain complex except for permutations of the basis elements.

Definition 3.3. Let (C, Ω) be a chain complex. For $0 \leq i$, the *i-skeleton* $\operatorname{sk}_i(C)$ is a finite subcomplex of C of order i whose chain modules are $\big(\operatorname{sk}_i(C)\big)_\nu = C_\nu$ for $0 \leq \nu \leq i$.

So a basis of $\mathrm{sk}_i(C)$ is $\bigcup\limits_{\nu=0}^{i} \Omega_\nu$.

In particular, if (C,Ω) is a finite chain complex of order d, then the d-skeleton $\mathrm{sk}_d(C)$ of C is equal to C: $\mathrm{sk}_d(C) = C$.

Definition 3.4. Let C_ν be a chain module with basis $\Omega_\nu = \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$ in a chain complex (C,Ω). For an element $x = \sum_{i=1}^{k_\nu} a_i e_i^\nu$, the *support* of x is the set of all basis elements e_i^ν with coefficient $a_i \neq 0$:

$$\mathrm{supp}(x) := \{e_i^\nu \mid a_i \neq 0\} \subset \Omega_\nu.$$

The *boundary* of x is the support of $\partial_\nu(x)$:

$$\mathrm{bd}(x) := \mathrm{supp}(\partial_\nu x).$$

Notice the two facts:

$$\mathrm{supp}(x) = \emptyset \iff x = 0;$$
$$\mathrm{bd}(x) = \emptyset \iff x \in \ker(\partial_\nu).$$

Definition 3.5. A finite chain complex (C,Ω) of order d is *pure* if any $e_j^\nu \in \Omega$ for $0 \leq \nu \leq d-1$ and $0 \leq j \leq k_\nu$ is contained in the boundary of some $e_\ell^{\nu+1}$, $1 \leq \ell \leq k_{\nu+1}$, i. e.

$$e_j^\nu \in \mathrm{bd}(e_\ell^{\nu+1}) = \mathrm{supp}(\partial_{\nu+1} e_\ell^{\nu+1}).$$

Definition 3.6. A basis element $e \in \Omega$ of a chain complex (C,Ω) is called *maximal* if it is not contained in the boundary of any other basis element.

Remark 3.7. If (C,Ω) is a finite chain complex of order d then all basis elements of Ω_d are maximal. Furthermore, if C is even a pure chain complex, the maximal basis elements are exactly the ones in Ω_d.

Now we introduce an analogue to a simplex in a simplicial complex:

Definition 3.8. Let (C,Ω) be a chain complex with basis Ω. For a basis element $e_j^\mu \in \Omega_\mu$ we denote the subcomplex of C with $(C_{e_j^\mu}, \Omega_{e_j^\mu})$ whose chain modules $\left(C_{e_j^\mu}\right)_\nu$ have the following bases $\left(\Omega_{e_j^\mu}\right)_\nu$:

- $\left(\Omega_{e_j^\mu}\right)_\nu = \emptyset \quad$ for $\nu \geq \mu + 1$;
- $\left(\Omega_{e_j^\mu}\right)_\mu = \{e_j^\mu\}$;
- $\left(\Omega_{e_j^\mu}\right)_\nu = \bigcup\limits_{e \in \left(\Omega_{e_j^\mu}\right)_{\nu+1}} \mathrm{bd}(e) \quad$ for $0 \leq \nu \leq \mu - 1$.

Remark 3.9. The subcomplex $C_{e_j^\mu}$ is a pure finite chain complex of order μ. In particular, $\left(\Omega_{e_j^\mu}\right)_{\mu-1} = \mathrm{bd}(e_j^\mu)$.

If there is a basis element $e_i^\lambda \in \Omega_\lambda$ so that $e_i^\lambda \in \Omega_{e_j^\mu} \cap \Omega_{e_\ell^\kappa}$ for $\lambda < \mu, \kappa$, then $\Omega_{e_i^\lambda} \subset \Omega_{e_j^\mu} \cap \Omega_{e_\ell^\kappa}$, i. e. the complex $C_{e_i^\lambda}$ is contained in the chain complex generated by $\Omega_{e_j^\mu} \cap \Omega_{e_\ell^\kappa}$.

3.2. Critical Basis Elements in Chain Complexes.

Definition 3.10. Let (C, Ω) be a chain complex over R. Let Γ be the set of all maximal basis elements of C:

$$\Gamma := \{e \in \Omega \mid e \notin \mathrm{bd}(f) \text{ for all } f \in \Omega\}.$$

Let each basis Ω_ν of a chain module C_ν have an ordering in which the elements of $\Omega_\nu \setminus \Gamma$ come first. A basis element $e_j^\nu \in (\Omega_\nu \cap \Gamma)$ with $j \geq 2$ is called:

- *critical* if there exist $a_i \in R$, $1 \leq i \leq j-1$, so that
$$\partial_\nu(e_j^\nu) = \sum_{i=1}^{j-1} a_i \partial_\nu(e_i^\nu);$$
- *precritical* if there exist $a_i \in R$, $1 \leq i \leq j$, $a_j \neq 0$, so that
$$a_j \partial_\nu(e_j^\nu) = \sum_{i=1}^{j-1} a_i \partial_\nu(e_i^\nu);$$
- *noncritical* if e_j^ν is neither critical nor precritical.

Remark 3.11. (1) A critical basis element corresponds to a *spanning simplex* in a simplicial complex (cf. Kozlov, 2008, page 212).
(2) A critical basis element is always precritical. Conversely, a precritical element e_j^ν is critical if the coefficient a_j is a unit in R. Hence, the terms *precritical* and *critical* coincide if the principal ring R is a field.
(3) $\mathrm{supp}\big(a_j \partial_\nu(e_j^\nu)\big) = \mathrm{supp}\big(\partial_\nu(e_j^\nu)\big)$ if $a_j \neq 0$.
(4) It is possible to change the ordering of a basis Ω_ν so that all noncritical basis elements come first.

In a pure chain complex of order d, the precritical elements in the chain module basis Ω_d can be seen as the generators of homology. If all basis elements of Ω_d are either noncritical or critical, we can name a basis of $H_d(C)$. We follow Björner (Björner, 1992, page 254) who has done this for the special case of simplicial complexes and generalize his proof to chain complexes.

To formulate a theorem for $d \geq 1$ we introduce a new notation. Let C_ν be a chain module generated by $\Omega_\nu := \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$. Consider $\rho \in C_\nu$, $\rho = \sum_{i=1}^{k_\nu} a_i e_i^\nu$. Then we denote the coefficient of e_i^ν with $\rho(e_i^\nu) := a_i$.

Remark 3.12. Consider a finite chain complex (C, Ω) of order 0 with basis $\Omega = \Omega_0 = \{e_1^0, \dots, e_{k_d}^0\}$. There are $(k_0 - 1)$ critical basis elements and $H_0(C) \cong R^{k_0}$.

Theorem 3.13. *Let (C, Ω) be a pure finite chain complex of order $d \geq 1$ and Ω_d be a basis of C_d with $k_d \geq 1$ elements. Let there be $n < k_d$ critical elements $g_1, \dots, g_n$ in Ω_d and all other basis elements be noncritical. Let Ω_d be ordered in such a way that the noncritical elements come first:*

$$\Omega_d = \{e_1, \dots, e_m, g_1, \dots, g_n\}, \qquad m + n = k_d.$$

Then the following holds:

(1) $H_d(C) \cong R^n$.
(2) *For $n \geq 1$, there exist unique cycles $\rho_1, \dots, \rho_n$ in $H_d(C) \cong \ker \partial_d$ so that $\rho_i(g_j) = \delta_{ij}$.*
(3) *For $n \geq 1$, $\{\rho_1, \dots, \rho_n\}$ is a basis of $H_d(C)$.*

Proof. We start with the case $n = 0$, i. e. Ω_d has only noncritical elements, so $\Omega_d = \{e_1, \dots, e_m\}$.

We assume: There exists an element $x \in C_d$ with $x = \sum_{i=1}^m a_i e_i \neq 0$, so that $\partial_d(x) = 0$.

As $x \neq 0$ there is some $a_i \neq 0$. We define $i_0 := \max\{i \leq m \mid a_i \neq 0\}$ and conclude $a_{i_0}\partial_d(e_{i_0}) = \sum_{i<i_0}(-a_i)\partial_d(e_i)$, so e_{i_0} is not noncritical. $\lightning$

Hence, $\ker \partial_d = 0$, i. e. $H_d(C) = 0$.

For $n \geq 1$, the first statement is a consequence of the second and third, so we start proving the second statement using induction. Having $n \geq 1$ critical elements, $\Omega_d = \{e_1, \dots, e_m, g_1, \dots, g_n\}$.

We consider the chain complex $\widehat{C} := \bigcup_{i=1}^m C_{e_i}$, whose chain modules are: $\widehat{C}_d = \langle e_1, \dots, e_m\rangle$, $\widehat{C}_\nu = C_\nu$ for $d - 1 \geq \nu \geq 0$. The chain complex $\widehat{C}$ is a pure finite subcomplex of C of order d without precritical elements, as the only precritical elements in Ω_d are $g_1, \dots, g_n$. So $H_d(\widehat{C}) = 0$.

As the elements g_i are all critical, there exists a $\widehat{\rho}_i \in \widehat{C}_d$ for every $1 \leq i \leq n$ so that $\partial_d(\widehat{\rho}_i) = \partial_d(g_i)$. Let $\rho_i := g_i - \widehat{\rho}_i$, then $\partial_d(\rho_i) = \partial_d(g_i) - \partial_d(\widehat{\rho}_i) = 0$; so $\rho_i \in \ker \partial_d \cong H_d(C)$.

Because $\widehat{\rho}_i \in \widehat{C}_d$ for every $1 \leq i \leq n$, it is $\widehat{\rho}_i = \sum_{\ell=1}^m a_\ell^i e_\ell$ with $a_\ell^i \in R$. We get: $\rho_i = g_i + \sum_{\ell=1}^m(-a_\ell^i)e_\ell$. So there is only g_i with coefficient 1 in ρ_i:

$$\rho_i(g_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore, we have proved the second statement up to uniqueness.

Consider $\sigma_i \in H_d(C) \cong \ker(\partial_d)$ so that $\sigma_i = g_i + \sum_{\ell=1}^m b_\ell^i e_\ell$. Then we get $\sigma_i(g_j) = \delta_{ij}$. We conclude $\sigma_i - \rho_i = \sum_{\ell=1}^m (b_\ell^i + a_\ell^i)e_\ell$, so $\sigma_i - \rho_i \in H_d(\widehat{C}) = 0$. It follows $\sigma_i = \rho_i$, so we have shown uniqueness.

Now we have to show that $\{\rho_1, \dots, \rho_n\}$ generates $H_d(C)$ and is linearly independent.

Let $\sum_{i=1}^n a_i\rho_i = 0$ with $a_i \in R$. Because $\rho_i = g_i - \widehat{\rho}_i$, we get:

$$0 = \sum_{i=1}^n a_i g_i - \underbrace{\sum_{i=1}^n a_i \widehat{\rho}_i}_{\in \widehat{C}_d = \langle e_1, \dots, e_m\rangle}.$$

As $\{e_1, \dots, e_m, g_1, \dots, g_n\}$ is a basis of C_d, we conclude $a_i = 0$ for all i, so the elements $\rho_1, \dots, \rho_n$ are independent.

We consider $\sigma \in H_d(C) \cong \ker(\partial_d) \subset C_d$ and define $\tau := \sigma - \sum_{i=1}^n \sigma(g_i)\rho_i$. For $1 \leq j \leq n$, the coefficient $\tau(g_j)$ of g_j in τ is:

$$\tau(g_j) = \sigma(g_j) - \sum_{i=1}^n \sigma(g_i)\underbrace{\rho_i(g_j)}_{=\delta_{ij}} = \sigma(g_j) - \sigma(g_j) = 0.$$

Hence, τ is a combination of $e_1, \dots, e_m$, and we conclude $\tau \in H_d(\widehat{C}) = 0$, as $\tau \in \ker \partial_d|_{\widehat{C}_d}$. So we get $\sigma = \sum_{i=1}^n \sigma(g_i)\rho_i$, i. e. $\{\rho_1, \dots, \rho_n\}$ generates $H_d(C)$.

Hence, $\{\rho_1, \dots, \rho_n\}$ is a basis of $H_d(C)$, and because of $\rho_i(g_j) = \delta_{ij}$ we get $H_d(C) \cong R^n$. $\square$

In general, there are also precritical elements which are not critical. In this case we only know $H_d(C) \cong R^n$, but we cannot name a basis.

Theorem 3.14. *Let (C, Ω) be a pure finite chain complex of order $d \geq 1$ and Ω_d be a basis of C_d with $k_d \geq 1$ elements. Let there be $n < k_d$ precritical elements $g_1, \ldots, g_n$ in Ω_d. Let Ω_d be ordered in such a way that the noncritical elements come first:*

$$\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}, \qquad m + n = k_d.$$

Then $H_d(C) \cong R^n$.

Proof. The case $n = 0$ is already proven. We consider the general case with $n \geq 1$ precritical elements, so $\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}$.

As above we consider the chain complex $\widehat{C} := \bigcup_{i=1}^{m} C_{e_i}$ which is a pure subcomplex of C of order d without precritical elements, hence $H_d(\widehat{C}) = 0$.

As the elements g_i are all precritical, there exist some $\widehat{\rho}_i \in \widehat{C}_d$ and $0 \neq a_i \in R$ for every $1 \leq i \leq n$ so that $\partial_d(\widehat{\rho}_i) = \partial_d(a_i g_i)$. We define $\rho_i := a_i g_i - \widehat{\rho}_i$ and conclude: $\partial_d(\rho_i) = \partial_d(a_i g_i) - \partial_d(\widehat{\rho}_i) = 0$. Hence $\rho_i \in \ker \partial_d \cong H_d(C)$.

We show that the elements $\rho_1, \ldots, \rho_n$ are linearly independent. Let $\sum_{i=1}^{n} c_i \rho_i = 0$ with $c_i \in R$. It is $\rho_i = a_i g_i - \widehat{\rho}_i$, so we get:

$$0 = \sum_{i=1}^{n} c_i a_i g_i - \underbrace{\sum_{i=1}^{n} c_i \widehat{\rho}_i}_{\in \widehat{C}_d = \langle e_1, \ldots, e_m \rangle}.$$

As $\{e_1, \ldots, e_m, g_1, \ldots, g_n\}$ is a basis of C_d, we conclude $c_i a_i = 0$ for all i, so $c_i = 0$. Hence the elements $\rho_1, \ldots, \rho_n$ are independent.

Recall the generating number $\operatorname{gen}_R M$ and the degree of freedom $\operatorname{dgf}_R M$ which we have introduced in Section 2 for finitely generated R-modules M. About the boundary mapping $\partial_d \colon C_d \to C_{d-1}$ we know due to Theorem 2.1:

$$\operatorname{dgf}_R(C_d) = \operatorname{dgf}_R(\ker \partial_d) + \operatorname{dgf}_R(\operatorname{im} \partial_d). \tag{1}$$

We have shown above that there are n independent elements in $\ker \partial_d$, hence $\operatorname{dgf}_R(\ker \partial_d) \geq n$.

C_d is generated by $(m + n)$ elements. The equality $\operatorname{gen}_R(C_d) = \operatorname{dgf}_R(C_d)$ implies $\operatorname{dgf}_R(C_d) = m + n$.

Consider now the subcomplex $\widehat{C}$. Because all g_i are precritical, $\operatorname{im} \partial_d \supset \operatorname{im} \partial_d|_{\widehat{C}_d}$. By Theorem 2.1 we get:

$$m = \operatorname{dgf}_R(\widehat{C}_d) = \operatorname{dgf}_R \underbrace{(\ker \partial_d|_{\widehat{C}_d})}_{=0} + \operatorname{dgf}_R(\operatorname{im} \partial_d|_{\widehat{C}_d}) \leq \operatorname{dgf}_R(\operatorname{im} \partial_d).$$

Applying Equation (1) we conclude:

$$\operatorname{dgf}_R(\ker \partial_d) = n \quad \text{and} \quad \operatorname{dgf}_R(\operatorname{im} \partial_d) = m.$$

According to Theorem 2.2, $\operatorname{gen}_R(\ker \partial_d) = n$, therefore $H_d(C) \cong R^n$. □

3.3. **Reduced Homology.** Notice that there are chain complexes for which the augmentation homomorphism ϵ must be 0. For example, consider the chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1\}$ and $\Omega_0 = \{e_1^0\}$ with $\partial_1(e_1^1) = e_1^0$.

If there is a basis element $e_j^0 \in \Omega_0$ which is not contained in the boundary of any basis element of Ω_1 there is always a mapping $\epsilon \neq 0$ by defining $\epsilon(e_j^0) = 1$. In particular, ϵ can be defined this way for every finite chain complex of order 0. For chain complexes of order $d \geq 1$ we treat a special case:

Lemma 3.15. *Let (C,Ω) be a chain complex of order $d \geq 1$. Let $\Omega_1 = \{e_1^1, \ldots, e_{k_1}^1\}$ and $\Omega_0 = \{e_1^0, \ldots, e_{k_0}^0\}$ be bases of the chain modules C_1 and C_0. Let ${}^{\#}\operatorname{bd}(x) \geq 2$ for every $x \in C_1 \setminus \ker \partial_1$. Then a R-linear mapping $\epsilon\colon C_0 \to R$ exists such that $\epsilon(e_\ell^0) \neq 0$ for all $1 \leq \ell \leq k_0$.*

Proof. For any e_ℓ^0 which is not contained in the boundary of some e_j^1 we define $\epsilon(e_\ell^0) = 1$. Hence, we assume without loss of generality that the 1-skeleton of C is pure. Let

$$\partial_1(e_i^1) = \sum_{\ell=1}^{k_0} a_{i\ell} e_\ell^0 \quad \text{for } 1 \leq i \leq k_1.$$

Because $\epsilon \circ \partial_1 = 0$ we have to solve the following system of linear equations to define ϵ:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k_0} \\ a_{21} & a_{22} & \cdots & a_{2k_0} \\ \vdots & \vdots & & \vdots \\ a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 k_0} \end{pmatrix} \begin{pmatrix} \epsilon(e_1^0) \\ \epsilon(e_2^0) \\ \vdots \\ \epsilon(e_{k_0}^0) \end{pmatrix} = 0.$$

We assume that we get a line with only one entry $a_{ij} \neq 0$, i. e. there is an element e_j^0 with $\epsilon(e_j^0) = 0$. Getting such a line means that there exists an element $x \in C_1$ so that ${}^{\#}\operatorname{bd}(x) = 1$ which is a contradiction!

Therefore, the system of linear equations has a solution $\big(\epsilon(e_1^0), \ldots, \epsilon(e_{k_0}^0)\big)$ with all $\epsilon(e_\ell^0) \neq 0$. $\square$

4. Acyclic Chain Complexes and Cones

4.1. **Terms and Definitions.** We define acyclic chain complexes in the same way as acyclic simplicial complexes.

Definition 4.1. A chain complex (C,Ω) over a principal ring R is *acyclic* if the following holds for the homology groups:

$$H_0(C) \cong R, \qquad H_\nu(C) = 0 \quad \text{for } \nu \geq 1.$$

Some special simplicial complexes are *cones*: A simplicial cone has a distinguished vertex v_0, and for any maximal simplex S in the complex (i. e. simplices which are not contained in any other) holds: S has exact one facet which does not contain the vertex v_0. For example, simplices themselves are cones.

To define the concept of a cone for chain complexes, we want to abandon the geometrical idea of an apex.

Definition 4.2. Let (C,Ω) be a finite chain complex of order d over a principal ring R. For $0 \leq \nu \leq d$ let Ω_ν be a basis of C_ν, $\Omega_\nu = \{e_1^\nu, \ldots, e_{k_\nu}^\nu\}$.

(C,Ω) is a *cone* if the following conditions hold:

(1) For every $\nu \in \{1, \ldots, d\}$ there is a nonempty subset $S_\nu \subset \Omega_\nu$ so that:
   (a) $\operatorname{supp}(\partial_\nu e_j^\nu) \not\subset \bigcup_{e_i^\nu \in S_\nu \setminus \{e_j^\nu\}} \operatorname{supp}(\partial_\nu e_i^\nu)$ for every $e_j^\nu \in S_\nu$;
   (b) for every $e_k^\nu \in \Omega_\nu \setminus S_\nu$ there is an element $\tau_k \in C_{\nu+1}$ so that
   $$\partial_{\nu+1}\tau_k = c e_k^\nu + r \qquad \text{with } c \text{ unit in } R \text{ and } r \in \langle S_\nu \rangle.$$
(2) ${}^{\#}\operatorname{supp}(\partial_1 \sigma) \geq 2$ for every $\sigma \in C_1 \setminus \ker \partial_1$.
(3) There is a subset $\{e\} = S_0 \subset \Omega_0$ with ${}^{\#}S_0 = 1$ so that:

For every $e_k^0 \in \Omega_0 \setminus S_0$ there is an element $\tau_k \in C_1$ so that

$$\partial_1 \tau_k = c e_k^\nu + c_0 e \qquad \text{with } c \text{ unit in } R \text{ and } c_0 \neq 0.$$

Remark 4.3. (1) $c_0 \neq 0$ in condition 3 follows from condition 2.
(2) Recall the set Γ from Definition 3.10. It is always $\Gamma \cap \Omega_\nu \subset S_\nu$. In particular, $\Omega_d = S_d$.
(3) $(\ker \partial_\nu) \cap \langle S_\nu \rangle = \{0\}$ for all $1 \leq \nu \leq d$ because of condition 1a.

Lemma 4.4. *A cone (C, Ω) is acyclic.*

Proof. If $d = 0$ (C finite complex of order 0), then $\Omega_0 = S_0$. So we get $H_\nu(C) = 0$ for $\nu \geq 1$ and $H_0(C) \cong R$ because of ${}^\# S_0 = 1$.

Consider now the case $d \geq 1$. At first we show: $\ker \partial_\nu = \operatorname{im} \partial_{\nu-1}$ for $\nu \geq 1$. If $\nu > d$, there is nothing to do.

As $S_d = \Omega_d$ we conclude $\ker \partial_d = 0$. Therefore, $H_d(C) = 0$.

For $1 \leq \nu \leq d-1$, consider an element $\sigma \in \ker \partial_\nu$:

$$\sigma = \sum_{i=1}^{k_\nu} a_i e_i^\nu = \sum_{e_i^\nu \in S_\nu} a_i e_i^\nu + \sum_{e_i^\nu \notin S_\nu} a_i e_i^\nu.$$

There exists some $\tau_i \in C_{\nu+1}$ for every $e_i^\nu \notin S_\nu$ so that $\partial_{\nu+1} \tau_i = c_i e_i^\nu + r_i$ with c_i unit in R and $r_i \in \langle S_\nu \rangle$, as postulated. It is $\partial_{\nu+1} \tau_i \in \ker \partial_\nu$ because $\partial_\nu \circ \partial_{\nu+1} = 0$. Therefore, we get:

$$\underbrace{\sigma - \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1}) \partial_{\nu+1} \tau_i}_{\in \ker \partial_\nu} = \underbrace{\sum_{e_i^\nu \in S_\nu} a_i e_i^\nu - \sum_{e_i^\nu \notin S_\nu} a_i c_i^{-1} r_i}_{\in \langle S_\nu \rangle}$$

As $(\ker \partial_\nu) \cap \langle S_\nu \rangle = \{0\}$, we conclude $\sigma - \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1}) \partial_{\nu+1} \tau_i = 0$. Hence

$$\sigma = \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1}) \partial_{\nu+1} \tau_i \in \operatorname{im} \partial_{\nu+1}.$$

So $\ker \partial_\nu \subset \operatorname{im} \partial_{\nu+1}$. Therefore, $\ker \partial_\nu = \operatorname{im} \partial_{\nu+1}$ and $H_\nu(C) = 0$ for $1 \leq \nu \leq d-1$.

Now we have to show $H_0(C) \cong R$. We know $\ker \partial_0 = C_0 = \langle \Omega_0 \rangle$ with $\Omega_0 = \{e_1^0, \ldots, e_{k_0}^0\}$.

Recall that ${}^\# \operatorname{supp}(\partial_1 \sigma) \geq 2$ for every $\sigma \in C_1 \setminus \ker \partial_1$. Hence, $\lambda e_i^0 \notin \operatorname{im} \partial_1$ for any $1 \leq i \leq k_0$ and every $0 \neq \lambda \in R$.

As ${}^\# S_0 = 1$ we have ${}^\# \Omega_0 \geq 1$. We look at two cases separately:

- ${}^\# \Omega_0 = 1$, so $\Omega_0 = \{e_1^0\}$. It is $\operatorname{im} \partial_1 = 0$ because of the cone condition 2. So we get:

$$H_0(C) = \ker \partial_0 / \operatorname{im} \partial_1 = \langle e_1^0 \rangle / 0 \cong R$$

- ${}^\# \Omega_0 \geq 2$, so $\Omega_0 \setminus S_0 \neq \emptyset$. Without loss of generality, let $S_0 = \{e_1^0\}$. For every e_i^0 with $2 \leq i \leq k_0$ there exists some $\tau_i \in C_1$ so that

$$\partial_1 \tau_i = c_i e_i^0 + d_i e_1^0 \qquad \text{with } c_i \text{ unit in } R \text{ and } d_i \neq 0.$$

Without loss of generality we can assume:

$$\partial_1 \tau_i = e_i^0 + d_i e_1^0 \qquad \text{with } d_i \neq 0.$$

These elements are all independent and in $(\operatorname{im}\partial_1)$. As $\lambda e_1^0 \notin \operatorname{im}\partial_1$ for every $0 \neq \lambda \in R$, we get:

$$\begin{aligned} H_0(C) &= \ker\partial_0 / \operatorname{im}\partial_1 \\ &= \langle e_1^0, e_i^0 + d_i e_1^0 \mid 2 \le i \le k_0\rangle / \langle e_i^0 + d_i e_1^0 \mid 2 \le i \le k_0\rangle \\ &\cong \langle e_1^0\rangle \cong R. \end{aligned}$$ □

For later purpose we have a look at a special chain complex (C,Ω) of order 1. Let its chain modules C_0 be finitely generated and C_1 generated by one element:

$$C_1 = \langle e_1^1\rangle, \qquad C_0 = \langle e_1^0, \dots, e_k^0\rangle \quad \text{with } \operatorname{bd}(e_1^1) = \{e_1^0, \dots, e_k^0\}.$$

Therefore, $C = C_{e_1^1}$. What is known about the cardinality of the basis Ω_0 if (C,Ω) is acyclic?

C_0 is generated by at least one element ($k \geq 1$) if C is acyclic. The image $\operatorname{im}\partial_1$ is a free submodule of C_0 generated by $\partial_1 e_1^1 = \sum_{i=1}^k a_i e_i^0$ with $a_i \neq 0$ for all i. We distinguish some cases:

$k = 1$: $C_0 = \langle e_1^0\rangle$, so $\partial_1 e_1^1 = a_1 e_1^0$ with $a_1 \neq 0$. It follows:

$$H_0(C) = C_0 / \operatorname{im}\partial_1 = \langle e_1^0\rangle / \langle a_1 e_1^0\rangle \ncong R \quad \text{for } a_1 \neq 0.$$

$k \geq 3$: $C_0 = \langle e_1^0, \dots, e_k^0\rangle$, $\operatorname{im}\partial_1 = \langle \sum_{i=1}^k a_i e_i^0\rangle$.

We assume $H_0(C) = C_0/\operatorname{im}\partial_1 \cong R$. Then any two nonzero elements $[e_\ell^0]$, $[e_j^0]$, $\ell \neq j$, of $C_0/\operatorname{im}\partial_1$ are not independent, i. e. there exist elements $x, y \in R \setminus \{0\}$ so that

$$x e_\ell^0 + y e_j^0 = r \sum_{i=1}^k a_i e_i^0 = \sum_{i=1}^k (r a_i) e_i^0 \in \operatorname{im}\partial_1.$$

As $\{e_1^0, \dots, e_k^0\}$ is a basis of C_0 and $a_i \neq 0$ for all i we conclude $r = 0$. So $x e_\ell^0 + y e_j^0 = 0$ ↯ – a contradiction to the independence of e_ℓ^0 and e_j^0 in C_0. Hence, $H_0(C) \ncong R$.

The case $k = 2$ remains. Indeed, it is possible to get $H_0(C) \cong R$ then. Assume $\partial_1 e_1^1 = a_1 e_1^0 + a_2 e_2^0$ with a_2 unit in R. Then, $a_2^{-1} a_1 e_1^0 + e_2^0 \in \operatorname{im}\partial_1$, and we get:

$$H_0(C) = \langle e_1^0, e_2^0\rangle / \langle a_2^{-1} a_1 e_1^0 + e_2^0\rangle \cong \langle e_1^0\rangle \cong R.$$

But it is not necessary that a_1 or a_2 in $\partial_1 e_1^1 = a_1 e_1^0 + a_2 e_2^0$ is a unit. Take $R = \mathbb{Z}$ and $\partial_1 e_1^1 = 2e_1^0 + 3e_2^0$. Then:

$$H_0(C) = \langle e_1^0, e_2^0\rangle / \langle 2e_1^0 + 3e_2^0\rangle \cong \mathbb{Z}$$

as this factor module is generated by $e_1^0 + e_2^0$.

We summarize:

Lemma 4.5. *Let (C,Ω) be a pure chain complex of order 1 over a principal ring R. Let its chain modules C_0 be finitely generated and C_1 generated by one element:*

$$C_1 = \langle e_1^1\rangle, \qquad C_0 = \langle e_1^0, \dots, e_k^0\rangle \quad \textit{with } \operatorname{bd}(e_1^1) = \{e_1^0, \dots, e_k^0\}.$$

If (C,Ω) is acyclic, then C_0 is generated by two elements (so $k = 2$).

The converse is not true. If C_0 is generated by two elements, then C is not necessarily acyclic. We take $R = \mathbb{Z}$ and $\partial_1 e_1^1 = 2e_1^0 + 2e_2^0$ and get:

$$H_0(C) = \langle e_1^0, e_2^0\rangle / \langle 2e_1^0 + 2e_2^0\rangle \ncong \mathbb{Z}$$

as the factor module is not torsion free: $2[e_1^0 + e_2^0] = [0|$.

We present a few examples of cones and acyclic chain complexes.

Examples 4.6. (1) *Consider a simplicial complex D which is a cone in the common sense. Then D is also a cone according to our definition: If the distinguished vertex of D is v_0, then we choose these basis elements for S_ν which correspond to ν-dimensional simplices containing the vertex v_0. In particular, $S_0 = \{v_0\}$.*

There is no need to set S_ν in this way, as the following example shows.

(2) *Consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let there be the following boundary mappings:*

$$\partial_2(e_1^2) = e_1^1 - e_2^1 + e_3^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_3^0 - e_2^0, \\ \partial_1(e_2^1) &= e_3^0 - e_1^0, \\ \partial_1(e_3^1) &= e_2^0 - e_1^0. \end{aligned}$$

We choose $S_2 = \Omega_2 = \{e_1^2\}$, $S_1 = \{e_1^1, e_2^1\}$ and $S_0 = \{e_1^0\}$. Then all cone conditions are fulfilled but $e_1^0 \notin C_{e_1^1}$.

(3) *Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$ so that:*

$$\partial_2(e_1^2) = e_1^1 + e_2^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_1^0 - e_2^0. \end{aligned}$$

The choice of $S_2 = \Omega_2 = \{e_1^2\}$, $S_1 = \{e_1^1\}$ and $S_0 = \{e_1^0\}$ makes (C, Ω) a cone. Notice that this chain complex does not come from a simplicial complex!

(4) *Again, we consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$. Let $\Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$, $\Omega_0 = \{e_1^0, e_2^0\}$ and:*

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_2(e_2^2) &= e_2^1 + e_3^1, & \partial_1(e_2^1) &= e_1^0 - e_2^0, \\ \partial_2(e_3^2) &= e_3^1 + e_4^1, & \partial_1(e_3^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_4^1) &= e_1^0 - e_2^0. \end{aligned}$$

We choose $S_2 = \Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $S_1 = \{e_2^1\}$ and $S_0 = \{e_1^0\}$. Because $\partial_2(e_2^2 - e_3^2) = e_2^1 - e_4^1$ all conditions for a cone are satisfied.

(5) *Our last example is also a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$. Let $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$, $\Omega_0 = \{e_1^0, e_2^0, e_3^0, e_4^0\}$ so that*

$$\partial_2(e_1^2) = e_1^1 + e_2^1 + e_3^1 + e_4^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_3^0 - e_2^0, \\ \partial_1(e_3^1) &= e_4^0 - e_3^0, \\ \partial_1(e_4^1) &= e_1^0 - e_4^0. \end{aligned}$$

The homology of C is

$$H_2(C) = 0, \quad H_1(C) = 0, \quad H_0(C) \cong \mathbb{Z},$$

hence C is acyclic. But C is not a cone, which we will see as follows:

By definition holds: $\mathrm{bd}(e_1^1) \cup \mathrm{bd}(e_3^1) = \Omega_0 = \mathrm{bd}(e_2^1) \cup \mathrm{bd}(e_4^1)$, and due to the cone condition 1a we conclude ${}^\# S_1 \leq 2$, hence ${}^\#(\Omega_1 \setminus S_1) \geq 2$. Because

$\mathrm{bd}(e_1^2) = \Omega_1$ *it follows* $^{\#}\big(\mathrm{bd}(e_1^2) \cap (\Omega_1 \setminus S_1)\big) \geq 2$, *and this is a contradiction to the cone condition 1b.*

5. Shellable Chain Complexes

In Björner *et al.* (1999, page 205), shellability is defined for *regular cell complexes* which are more general than simplicial complexes. In a similar way, we define shellability of chain complexes.

5.1. Definition and First Examples.

Let (C, Ω) be a chain complex with basis Ω. We define a mapping:

$$s\colon \Omega \to \mathbb{Z}; \qquad e_i^\nu \mapsto \nu = \text{order of the complex } C_{e_i^\nu}.$$

Definition 5.1. Let (C, Ω) be a finite chain complex of order d over a principal ring R. Let $\Gamma := \{e \in \Omega \mid e \notin \mathrm{bd}(f) \text{ for all } f \in \Omega\} \neq \emptyset$ be the set of all maximal basis elements of C. An order of the basis elements in $\Gamma := \{g_1, \ldots, g_k\}$ is a *shelling* (or a *shelling order*) if $d = 0$ or if the following holds for $d \geq 1$:

(1) For $2 \leq j \leq k$, the set
$$\Omega_{g_j} \cap \Big(\bigcup_{i=1}^{j-1} \Omega_{g_i}\Big)$$
generates a pure chain complex of order $s(g_j) - 1$.

(2) For $2 \leq j \leq k$, the set $(\Omega_{g_j})_{s(g_j)-1}$ has a shelling in which the basis elements of $\Big(\Omega_{g_j} \cap (\bigcup_{i=1}^{j-1} \Omega_{g_i})\Big)_{s(g_j)-1}$ come first.

(3) $(\Omega_{g_1})_{s(g_1)-1}$ has a shelling.

Then, the chain complex (C, Ω) is *shellable*.

Remark 5.2.

(1) It must be $s(g_1) = d$, otherwise it would be impossible to get a shelling because of condition 1. So we can rewrite condition 3 as follows: $(\Omega_{g_1})_{d-1}$ has a shelling.

(2) The definition of shellability for simplicial complexes contains only condition 1 (cf. Kozlov, 2008, ch. 12). As the boundary of a simplex is always shellable, the conditions 2 and 3 are trivially satisfied. Hence, the definition above contains shellability of simplicial complexes.

(3) If $\Gamma = \{g_1\}$, only the third condition is relevant.

(4) If (C, Ω) is a shellable chain complex of order d, the chain modules of C are $C_\nu \neq 0$ for $0 \leq \nu \leq d$.

(5) In a shellable chain complex (C, Ω), each subcomplex $C_{e_i^\nu}$ is shellable, due to the conditions 2 and 3.

(6) It follows from the conditions 2 and 3 as well that the $\big(s(g_j) - 1\big)$-skeleton of C_{g_j} is shellable for $1 \leq j \leq k$.

(7) For a precritical element $g_j \in \Gamma$ it is
$$\Omega_{g_j} \cap \Big(\bigcup_{i=1}^{j-1} \Omega_{g_i}\Big) = \Omega_{g_j} \setminus \{g_j\}.$$
Therefore, it is possible to rearrange the elements in a shelling of Γ so that all precritical elements come at last.

In opposite to shellable simplicial complexes the homology of shellable chain complexes is not clear. Consider the following examples:

Examples 5.3. (1) *Let (C, Ω) be a chain complex of order 1 over $\mathbb{Z}$ so that $C_1 = \langle e_1^1 \rangle$, $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$ for some $k \geq 1$ and $\partial_1 e_1^1 = \sum_{i=1}^k e_i^0$. This complex is shellable and its homology is:*

$$H_1(C) = 0, \qquad H_0(C) \cong \mathbb{Z}^{k-1}.$$

If $k = 2$, this chain complex is acyclic and even a cone.

(2) *Let (C, Ω) be a chain complex of order 1 over $\mathbb{Z}$ so that $C_1 = \langle e_1^1, e_2^1 \rangle$ and $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$ for some $k \geq 1$. We assume: $\partial_1 e_1^1 = \partial_1 e_2^1 = \sum_{i=1}^k e_i^0$. This complex is shellable and its homology is:*

$$H_1(C) \cong \mathbb{Z}, \qquad H_0(C) \cong \mathbb{Z}^{k-1}.$$

(3) *Let (C, Ω) be a chain complex of order 1 over $\mathbb{Z}$ so that $C_1 = \langle e_1^1, e_2^1 \rangle$ and $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$ for some $k \geq 2$. We assume: $\partial_1 e_1^1 = \sum_{i=1}^k e_i^0$ and $\partial_1 e_2^1 = -e_1^0 + \sum_{i=2}^k e_i^0$. This complex is shellable. About the homology we know:*

$$H_1(C) = 0, \qquad H_0(C) \ncong \mathbb{Z}^i \quad \textit{for any } i \in \mathbb{N}$$

as there are torsion elements in $H_0(C)$, for example e_1^0.

So we need more conditions on shellable chain complexes to get some information about homology. We will treat this later in Section 6.

5.2. **Monotonically Descending Shellings.** Because of the first shelling condition the set $\left(\Omega_{g_j} \cap \left(\bigcup_{i=1}^{j-1} \Omega_{g_i}\right)\right)$ generates a pure chain complex of order $(s(g_j) - 1)$ for $2 \geq j \geq k$. We ask: Is it always possible to get a shelling of Γ so that $s(g_i) \geq s(g_{i+1})$ for all i? Indeed, this is true, as we will show in this section.

Definition 5.4. Let (C, Ω) be a shellable chain complex over R, finite of order d, and $\Gamma \subset \Omega$ be the subset of all maximal basis elements. A shelling of $\Gamma := \{g_1, \ldots, g_k\}$ is *monotonically descending* if $s(g_i) \geq s(g_{i+1})$ for all $1 \leq i \leq k-1$.

A *failure* in a shelling of $\Gamma = \{g_1, \ldots, g_k\}$ is a pair (i, j) with $i < j$ and $s(g_i) < s(g_j)$.

Therefore, a monotonically descending shelling is a shelling without failures.

Remark 5.5. If (C, Ω) is a shellable chain complex and *pure*, then every shelling of Γ is monotonically descending.

At first, we will prove the following lemma:

Lemma 5.6. *Let (C, Ω) and Γ be as in Definition 5.4. Let there be $m \geq 1$ failures in a shelling of Γ. Then it is possible to permute the elements of Γ so that there is a new shelling of Γ with $(m-1)$ failures.*

Proof. Let $\Gamma := \{g_1, \ldots, g_k\}$, ordered in a shelling. As (C, Ω) is shellable we know $s(g_1) \geq s(g_i)$ for any $2 \leq i \leq k$.

There is a minimal $2 \leq i_0 \leq k-1$ so that $s(g_{i_0}) < s(g_{i_0+1})$. We want to show that we still have a shelling after permuting g_{i_0} and g_{i_0+1}, i.e. the ordered set $\{g_1, \ldots, g_{i_0-1}, g_{i_0+1}, g_{i_0}, \ldots, g_k\}$ is also a shelling.

At first, we consider the chain complex generated by:

$$\Delta = \left(\bigcup_{i=1}^{i_0} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}}.$$

About this complex we know:

- It is a pure chain complex of order $(s(g_{i_0+1})-1)$. Hence, all maximal basis elements in Δ are in $\left(\Omega_{g_{i_0+1}}\right)_{s(g_{i_0+1})-1}$.
- $\left(\Omega_{g_{i_0+1}}\right)_{s(g_{i_0+1})-1}$ has a shelling in which the basis elements from Δ come first.

We divide the intersection into two parts:

$$\Delta = \left(\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}}\right)\cup\Big(\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}\Big).$$

Because $s(g_{i_0}) < s(g_{i_0+1})$, the set $\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}$ generates a chain complex of order $t \le s(g_{i_0})-1 \le s(g_{i_0+1})-2$. Therefore, any maximal basis element e in $\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}$ is contained in the boundary of some other basis element $f\in\Delta$, otherwise Δ would not generate a pure chain complex. Because e is maximal in $\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}$ we conclude: $f\notin\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}$. Hence, $f\in\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}}$. Therefore we conclude:

$$\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}\subset\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}},\quad \text{hence } \Delta=\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}}.$$

Hence, the chain complex generated by $\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}}$ is pure of order $s(g_{i_0+1})-1$ and satisfies all other shelling properties, too.

We consider now the chain complex with basis

$$\Lambda=\left(\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cup\Omega_{g_{i_0+1}}\right)\cap\Omega_{g_{i_0}}=\left(\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0}}\right)\cup\Big(\Omega_{g_{i_0+1}}\cap\Omega_{g_{i_0}}\Big).$$

Above we have shown $\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}\subset\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0+1}}\subset\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)$. As $\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}\subset\Omega_{g_{i_0}}$ we conclude:

$$\Omega_{g_{i_0}}\cap\Omega_{g_{i_0+1}}\subset\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0}},\quad \text{hence } \Lambda=\Big(\bigcup_{i=1}^{i_0-1}\Omega_{g_i}\Big)\cap\Omega_{g_{i_0}}.$$

Because the g_i are ordered in a shelling, Λ generates a pure chain complex of order $(s(g_{i_0})-1)$ which also satisfies all other shelling properties.

Therefore, $\{g_1,\dots,g_{i_0-1},g_{i_0+1},g_{i_0},\dots,g_k\}$ is a shelling order with exactly one failure less. □

By repeated application of this lemma we get:

Theorem 5.7. *Let (C,Ω) be a shellable chain complex over R, finite of order d, and $\Gamma\subset\Omega$ be the subset of all maximal basis elements. Then a monotonically descending shelling of $\Gamma=\{g_1,\dots,g_k\}$ exists.*

5.3. ***i*-Skeletons of Shellable Chain Complexes.** In Section 3.1 we introduced i-skeletons $\mathrm{sk}_i(C)$ of a chain complex (C,Ω) as subcomplexes whose chain modules $(\mathrm{sk}_i(C))_\nu$ are zero for $\nu>i$ and equal to C_ν otherwise.

Lemma 5.8. *Let (C,Ω) be a pure shellable chain complex, finite of order $d\ge 1$. The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C is shellable, too.*

Proof. Let $\Omega_d = \{e_1^d, \dots, e_{k_d}^d\}$. We have to show: $\Omega_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$ has a shelling. We will do this by an inductive argument.

By definition we know that $\mathrm{bd}(e_1^d) = (\Omega_{e_1^d})_{d-1}$ has a shelling.

Consider now $\bigcup_{i=1}^{\ell} \mathrm{bd}(e_i^d) \subset \Omega_{d-1}$ for $2 \leq \ell \leq k_d$. We assume that the set $\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)$ has a shelling, and we want to show that

$$\Big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\Big) \cup (\Omega_{e_\ell^d})_{d-1} = \Big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\Big) \cup \mathrm{bd}(e_\ell^d)$$

has a shelling, too.

Because the complex C is shellable we know that $(\Omega_{e_\ell^d})_{d-1}$ has a shelling in which the elements of

$$\Big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\Big) \cap \Omega_{e_\ell^d} = \left(\Big(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\Big) \cap \Omega_{e_\ell^d}\right)_{d-1}$$

come first. If e_ℓ^d is precritical, then $\big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\big) \cap \Omega_{e_\ell^d} = \big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\big)$, and we are done.

So we assume e_ℓ^d noncritical. Then let $(\Omega_{e_\ell^d})_{d-1} = \{f_1, \dots, f_s, h_1, \dots, h_t\}$ be a shelling so that $\big(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\big) \cap \Omega_{e_\ell^d} = \{f_1, \dots, f_s\}$ and $s \geq 1$. As e_ℓ^d is noncritical we also have $t \geq 1$.

Let $\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d) = \{e_1, \dots, e_r, f_1, \dots, f_s\}$. This order is not necessarily a shelling but this does not matter.

We want to show that $\{e_1, \dots, e_r, f_1, \dots, f_s, h_1, \dots, h_t\}$ has a shelling. We start by checking

$$\left(\Big(\bigcup_{i=1}^{r} \Omega_{e_i}\Big) \cup \Big(\bigcup_{i=1}^{s} \Omega_{f_i}\Big)\right) \cap \Omega_{h_1} = \left(\Big(\bigcup_{i=1}^{r} \Omega_{e_i}\Big) \cap \Omega_{h_1}\right) \cup \left(\Big(\bigcup_{i=1}^{s} \Omega_{f_i}\Big) \cap \Omega_{h_1}\right).$$

As $(\Omega_{e_\ell^d})_{d-1} = \{f_1, \dots, f_s, h_1, \dots, h_t\}$ is a shelling the set $\big(\bigcup_{i=1}^{s} \Omega_{f_i}\big) \cap \Omega_{h_1}$ generates a pure chain complex of order $(d-2)$ and satisfies all shelling conditions.

Because $\big(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\big) \cap \Omega_{e_\ell^d} = \bigcup_{j=1}^{s} \Omega_{f_j}$ we get $\big(\bigcup_{i=1}^{r} \Omega_{e_i}\big) \cap \Omega_{h_1} \subset \bigcup_{j=1}^{s} \Omega_{f_j}$ and therefore $\big(\bigcup_{i=1}^{r} \Omega_{e_i}\big) \cap \Omega_{h_1} \subset \bigcup_{j=1}^{s} \Omega_{f_j} \cap \Omega_{h_1}$. We conclude:

$$\left(\Big(\bigcup_{i=1}^{r} \Omega_{e_i}\Big) \cup \Big(\bigcup_{i=1}^{s} \Omega_{f_i}\Big)\right) \cap \Omega_{h_1} = \Big(\bigcup_{i=1}^{s} \Omega_{f_i}\Big) \cap \Omega_{h_1},$$

so the set $\{e_1, \dots, e_r, f_1, \dots, f_s, h_1\}$ has a shelling.

For any $1 \leq j \leq t-1$ we assume the set $\{e_1, \dots, e_r, f_1, \dots, f_s, h_1, \dots, h_j\}$ has a shelling, i. e. the set

$$\Big(\bigcup_{i=1}^{r} \Omega_{e_i}\Big) \cup \Big(\bigcup_{i=1}^{s} \Omega_{f_i}\Big) \cup \Big(\bigcup_{i=1}^{j} \Omega_{h_i}\Big)$$

generates a shellable chain complex of order $(d-1)$. Then a similar argument as above shows that

$$\begin{aligned}
&\left(\left(\bigcup_{i=1}^{r}\Omega_{e_i}\right)\cup\left(\bigcup_{i=1}^{s}\Omega_{f_i}\right)\cup\left(\bigcup_{i=1}^{j}\Omega_{h_i}\right)\right)\cap\Omega_{h_{j+1}}\\
&=\underbrace{\left(\left(\bigcup_{i=1}^{r}\Omega_{e_i}\right)\cap\Omega_{h_{j+1}}\right)}_{\subset\left(\bigcup_{i=1}^{s}\Omega_{f_i}\right)\cap\Omega_{h_{j+1}}}\cup\underbrace{\left(\left(\left(\bigcup_{i=1}^{s}\Omega_{f_i}\right)\cup\left(\bigcup_{i=1}^{j}\Omega_{h_i}\right)\right)\cap\Omega_{h_{j+1}}\right)}_{\text{generates a pure complex of order } (d-2)}\\
&=\left(\left(\bigcup_{i=1}^{s}\Omega_{f_i}\right)\cup\left(\bigcup_{i=1}^{j}\Omega_{h_i}\right)\right)\cap\Omega_{h_{j+1}}.
\end{aligned}$$

This set is a basis of a shellable pure chain complex of order $(d-2)$. Therefore, the set $\{e_1,\dots,e_r,f_1,\dots,f_s,h_1,\dots,h_{j+1}\}$ has a shelling.

By induction we conclude that the set

$$\{e_1,\dots,e_r,f_1,\dots,f_s,h_1,\dots,h_t\}=\bigcup_{i=1}^{\ell}\mathrm{bd}(e_i^d)$$

has a shelling. Therefore, the first induction delivers: $\Omega_{d-1}=\bigcup_{i=1}^{k_d}\mathrm{bd}(e_i^d)$ has a shelling. $\square$

Theorem 5.9. *Let (C,Ω) be a shellable chain complex, finite of order $d\geq 1$. The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C is shellable, too.*

Proof. We have already proved this statement if (C,Ω) is pure. So we have only to consider the non-pure case.

Let $\Gamma\subset\Omega$ be the set of all maximal basis elements and let the elements of Γ be ordered in a monotonically descending shelling.

Let $\Omega_d=\{e_1^d,\dots,e_{k_d}^d\}\subset\Gamma$, and $\Gamma\cap\Omega_{d-1}=\{g_1^{d-1},\dots,g_m^{d-1}\}$. In the chosen shelling order of Γ the elements of Ω_d come first, followed by all elements of $\Gamma\cap\Omega_{d-1}$. The basis of the chain module $\big(\mathrm{sk}_{d-1}(C)\big)_{d-1}$ is

$$\Omega_{d-1}=\left(\bigcup_{i=1}^{k_d}\mathrm{bd}(e_i^d)\right)\cup\{g_1^{d-1},\dots,g_m^{d-1}\}.$$

We have already proved that $\left(\bigcup_{i=1}^{k_d}\mathrm{bd}(e_i^d)\right)$ has a shelling because the subcomplex generated by $\bigcup_{i=1}^{k_d}\Omega_{e_i^d}$ is shellable and pure.

Let $\widehat{\Omega}_{e_i^d}:=\Omega_{e_i^d}\setminus\{e_i^d\}$ be the basis of the $(d-1)$-skeleton of $C_{e_i^d}$. Then we get:

$$\left(\bigcup_{i=1}^{k_d}\Omega_{e_i^d}\right)\cap\Omega_{g_1^{d-1}}=\left(\bigcup_{i=1}^{k_d}\widehat{\Omega}_{e_i^d}\right)\cap\Omega_{g_1^{d-1}}.$$

As C is shellable, the set on the left side satisfies all shelling conditions. Therefore, $\left(\bigcup_{i=1}^{k_d}\mathrm{bd}(e_i^d)\right)\cup\{g_1^{d-1}\}\subset\Omega_{d-1}$ has a shelling, and by induction we conclude that Ω_{d-1} has a shelling, too. $\square$

Corollary 5.10. *Let (C,Ω) be a shellable chain complex of order d. For $0\leq i\leq d$, every i-skeleton $\mathrm{sk}_i(C)$ of C is shellable.*

Hence, if any i-skeleton $\mathrm{sk}_i(C)$ of a chain complex C is not shellable, the complex itself is not shellable.

In the proof of Lemma 5.8 we used a special ordering of the chain module bases Ω_ν. We emphasise it for later purpose:

Remark 5.11. Let (C, Ω) be a shellable chain complex of order d and let its set Γ of maximal basis elements be ordered in a monotonically descending shelling (cf. Definition 5.4). Then, we can order the elements in the bases Ω_ν of the chain modules C_ν, $0 \leq \nu \leq d-1$, as follows:

The first segment in Ω_ν are the elements of $\mathrm{bd}(e_1^{\nu+1})$, ordered in a shelling. Then we add the elements of $\mathrm{bd}(e_2^{\nu+1}) \setminus \mathrm{bd}(e_1^{\nu+1})$ in the same order as in the shelling of $\mathrm{bd}(e_2^{\nu+1})$. We proceed iteratively, adding the elements of $\mathrm{bd}(e_i^{\nu+1})$ which are not contained in any $\mathrm{bd}(e_\ell^{\nu+1})$ for $\ell < i$. Eventually we are left with the elements of $\Omega_\nu \cap \Gamma$, which we will add in the same order as they occur in Γ. As proven above, this ordering delivers a shelling of Ω_ν.

6. Regular Chain Complexes

6.1. **Definition and Examples.**

Definition 6.1. Let (C, Ω) be a shellable chain complex of order d over a principal ring R and $\Gamma \subset \Omega$ be the set of all maximal basis elements.

Let $\Omega_d := \{e_1^d, \dots, e_{k_d}^d\}$ and $\Omega_\nu := \{e_1^\nu, \dots, e_{m_\nu}^\nu, e_{m_\nu+1}^\nu, \dots, e_{k_\nu}^\nu\}$ for $\nu \leq d-1$ so that the following holds[1]:

- $\Gamma \cap \Omega_\nu = \{e_{m_\nu+1}^\nu, \dots, e_{k_\nu}^\nu\}$;
- $\Gamma = \{e_1^d, \dots, e_{k_d}^d\} \cup \{e_{m_{d-1}+1}^{d-1}, \dots, e_{k_{d-1}}^{d-1}\} \cup \dots \cup \{e_{m_0+1}^0, \dots, e_{k_0}^0\}$ is a monotonically descending shelling;
- for $0 \leq \nu \leq d-1$, let each basis Ω_ν be ordered as in Remark 5.11.

Then, Γ has a *regular order* if the following two conditions are fulfilled:

(1) For any $e_\ell^\nu \in \Gamma$ (i. e. $m_\nu + 1 \leq \ell \leq k_\nu$):
If $\mathrm{bd}(e_\ell^\nu) \subset \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^\nu)$, then e_ℓ^ν is precritical, i. e. there exist elements $a_i \in R$, $1 \leq i \leq \ell$, $a_\ell \neq 0$, so that
$$a_\ell \partial_\nu(e_\ell^\nu) = \sum_{i=1}^{\ell-1} a_i \partial_\nu(e_i^\nu).$$

(2) For any $e_\ell^\nu \in \Omega_\nu$, $\nu \geq 1$, let $(\Omega_{e_\ell^\nu})_{\nu-1} = \mathrm{bd}(e_\ell^\nu) := \{f_1^{\nu-1}, \dots, f_{n_\ell}^{\nu-1}\}$ be a shelling in which the elements of $(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^\nu}) \cap \Omega_{e_\ell^\nu}$ come first so that for any $1 \leq j \leq n_\ell$ holds:
If $\mathrm{bd}(f_j^{\nu-1}) \subset \bigcup_{i=1}^{j-1} \mathrm{bd}(f_i^{\nu-1})$, then $c_j \partial_{\nu-1}(f_j^{\nu-1})$ is a linear combination of $\partial_{\nu-1}(f_i^{\nu-1})$ for some $0 \neq c_j \in R$, i. e. there exist $c_i \in R$, $1 \leq i \leq j$, $c_j \neq 0$, so that
$$c_j \partial_{\nu-1}(f_j^{\nu-1}) = \sum_{i=1}^{j-1} c_i \partial_{\nu-1}(f_i^{\nu-1}).$$

The chain complex (C, Ω) is *regular* if the set Γ has a regular order.

---

[1]Notice two special cases:

- $m_\nu = k_\nu$. Then $\Gamma \cap \Omega_\nu = \emptyset$.
- $m_\nu = 0$. Then $\Gamma \cap \Omega_\nu = \Omega_\nu$. This is always valid for Ω_d.

Definition 6.2. A regular chain complex (C, Ω) whose subcomplexes $C_{e_i^\nu}$ are all acyclic is called *totally regular*.

Remark 6.3. A chain complex which comes from a simplicial complex is always totally regular. This is caused by the geometry of a simplex and the special boundary mapping.

Any finite chain complex of order 0 satisfies trivially all regularity conditions, so it is totally regular. But there also exist more serious examples:

Examples 6.4. (1) *Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_1^0 + 2e_2^0$. The reader may convince himself that this chain complex is shellable.*

It is $\mathrm{bd}(e_1^1) = \Omega_0 = \mathrm{bd}(e_2^1)$, but $\partial_1(e_1^1)$ and $\partial_1(e_2^1)$ are linearly independent. So (C, Ω) is not regular.

Furthermore, it is $\partial_1(2e_1^1 - e_2^1) = 3e_1^0$. As $e_1^0 \notin \mathrm{im}\,\partial_1$, the factor module $H_0(C) = C_0/\,\mathrm{im}\,\partial_1$ is not torsion free. Hence, $H_0(C) \ncong \mathbb{Z}$, i.e. the chain complex (C, Ω) is not acyclic.

(2) *We consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ whose chain modules have the following bases: $\Omega_2 = \{e_1^2, e_2^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let the boundary mappings ∂_2 and ∂_1 defined by:*

$$\begin{aligned} \partial_2(e_1^2) &= 2e_1^1 + e_2^1 + e_3^1, & \qquad \partial_1(e_1^1) &= e_1^0 - e_2^0, \\ \partial_2(e_2^2) &= e_1^1 + e_2^1, & \partial_1(e_2^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_3^1) &= e_2^0 - e_1^0. \end{aligned}$$

This chain complex is shellable. But its natural order is not regular because $\mathrm{bd}(e_2^2) \subset \mathrm{bd}(e_1^2)$. Changing the order of e_1^2 and e_2^2 delivers a regular order, indeed! Therefore, this chain complex is regular, but not totally regular as $C_{e_1^2}$ is not acyclic.

Now we have a look at the homology of (C, Ω).

- $\ker \partial_2 = 0$, *so* $H_2(C) = 0$.
- *It is* $\ker \partial_1 = \langle (e_1^1 + e_2^1), (e_1^1 + e_3^1) \rangle = \mathrm{im}\,\partial_2$, *hence* $H_1(C) = 0$.
- *As* $\ker \partial_0 = \langle e_1^0, e_2^0 \rangle$ *and* $\mathrm{im}\,\partial_1 = \langle e_1^0 - e_2^0 \rangle$ *we get* $H_0(C) \cong \mathbb{Z}$.

Therefore, (C, Ω) is acyclic. It is even a cone if we choose $S_2 = \{e_1^2, e_2^2\}$, $S_1 = \{e_1^1\}$ and $S_0 = \{e_1^0\}$.

(3) *Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_1^0 + e_2^0$.*

This complex is shellable, and every subcomplex $C_{e_j^\nu}$ is acyclic. Furthermore, we have $\mathrm{bd}(e_1^1) = \{e_1^0, e_2^0\} = \mathrm{bd}(e_2^1)$, but $\partial_1(e_1^1)$ and $\partial_1(e_2^1)$ are linearly independent. So (C, Ω) is not regular.

We compute the homology groups:

- $\ker \partial_1 = 0$, *so* $H_1(C) = 0$.
- $\ker \partial_0 = \langle e_1^0, e_2^0, e_3^0 \rangle$ *and* $\mathrm{im}\,\partial_1 = \langle e_1^0 + e_2^0, 2e_1^0 + e_2^0 \rangle$, *therefore we get* $H_0(C) \cong \mathbb{Z}$.

Hence, this chain complex is acyclic. But it is not a cone as $\partial_1(e_1^1 - e_2^1) = e_1^0$.

(4) *Let (C,Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ and $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$, $\Omega_0 = \{e_1^0, e_2^0\}$ be the bases of its chain modules. Let*

$$\partial_2(e_1^2) = e_1^1 + e_2^1 + e_3^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_1^0 - e_2^0, \\ \partial_1(e_2^1) &= e_1^0 - e_2^0, \\ \partial_1(e_3^1) &= 2(e_2^0 - e_1^0). \end{aligned}$$

This chain complex is shellable and regular.

We compute its homology groups:

- $\ker \partial_2 = 0$, *so* $H_2(C) = 0$.
- *As* $\ker \partial_1 = \langle (e_1^1 - e_2^1), (e_1^1 + e_2^1 + e_3^1) \rangle$ *and* $\operatorname{im} \partial_2 = \langle e_1^1 + e_2^1 + e_3^1 \rangle$ *we get* $H_1(C) \cong \mathbb{Z}$.
- $\ker \partial_0 = \langle e_1^0, e_2^0 \rangle$ *and* $\operatorname{im} \partial_1 = \langle e_1^0 - e_2^0 \rangle$, *hence* $H_0(C) \cong \mathbb{Z}$.

Hence, (C,Ω) is not acyclic. In particular, (C,Ω) is not totally regular because then $C_{e_1^2} = C$ must be acyclic.

(5) *Let (C,Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let*

$$\begin{aligned} \partial_1(e_1^1) &= e_1^0, \\ \partial_1(e_2^1) &= e_1^0 + e_2^0 + e_3^0. \end{aligned}$$

We observe that this chain complex is acyclic, shellable and regular, but not a cone. As both subcomplexes $C_{e_1^1}$ and $C_{e_2^1}$ are not acyclic this chain complex is not totally regular.

If we change the order of the basis elements e_1^1 and e_2^1, we get an ordering which is not regular.

(6) *We consider an example for a chain complex cone again (cf. Examples 4.6 (4)). Let (C,Ω) be a finite chain complex of order 2 over $\mathbb{Z}$. Let $\Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$, $\Omega_0 = \{e_1^0, e_2^0\}$ and:*

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, \\ \partial_2(e_2^2) &= e_2^1 + e_3^1, \\ \partial_2(e_3^2) &= e_3^1 + e_4^1, \end{aligned} \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_1^0 - e_2^0, \\ \partial_1(e_3^1) &= e_2^0 - e_1^0, \\ \partial_1(e_4^1) &= e_1^0 - e_2^0. \end{aligned}$$

This chain complex is shellable and regular. As every subcomplex $C_{e_j^\nu}$ is acyclic we conclude that this chain complex is even totally regular.

The reader may notice that this chain complex does not come from a simplicial complex!

(7) *We have a look at another example for a chain complex cone (cf. Examples 4.6 (3)). Let (C,Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$ so that:*

$$\partial_2(e_1^2) = e_1^1 + e_2^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_1^0 - e_2^0. \end{aligned}$$

As above, this chain complex is shellable, regular and every subcomplex $C_{e_j^\nu}$ is acyclic. So this chain complex is totally regular, too. It also does not come from a simplicial complex.

(8) *We modify our last example in some detail. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2, e_2^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$ so that:*

$$\partial_2(e_1^2) = e_1^1 + e_2^1, \qquad \partial_1(e_1^1) = e_2^0 - e_1^0,$$
$$\partial_2(e_2^2) = e_1^1 + e_2^1, \qquad \partial_1(e_2^1) = e_1^0 - e_2^0.$$

This chain complex is still totally regular, but it refuses to be acyclic as there is a critical basis element e_2^2.

Remark 6.5. In Section 5.1 we considered some examples of shellable chain complexes of order 1. The chain complexes in the Examples 5.3 (1) and (2) are all regular, but the complexes in the Example 5.3 (3) are not. Hence, the homology of regular chain complexes is not clear in general.

6.2. ***i*-Skeletons of Regular Chain Complexes.**

Lemma 6.6. *Let (C, Ω) be a pure regular chain complex, finite of order $d \geq 1$. The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C is regular, too. If (C, Ω) is even totally regular, then $\mathrm{sk}_{d-1}(C)$ is also totally regular.*

Proof. As C is a pure chain complex we have $\Gamma = \Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$ in a regular order.

The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ is a pure subcomplex of C of order $(d-1)$ whose basis is $\Omega \setminus \Omega_d$. We have to show that $\widehat{\Gamma} := \Omega_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$ has a regular order.

From Lemma 5.8 we know that the subcomplex $\mathrm{sk}_{d-1}(C)$ is shellable. Furthermore, $\mathrm{sk}_{d-1}(C)$ satisfies the second regularity condition as this property transmits from C. So, we have only to check the first regularity condition which we will do by induction.

At first, we consider the subcomplex of $\mathrm{sk}_{d-1}(C)$ which is generated by $\widehat{\Omega}_{e_1^d} := \Omega_{e_1^d} \setminus \{e_1^d\}$. We have $(\widehat{\Omega}_{e_1^d})_{d-1} = \mathrm{bd}(e_1^d)$. As C is regular there exists a shelling of $\mathrm{bd}(e_1^d)$ so that the second regularity condition is fulfilled. Hence, the elements in $(\widehat{\Omega}_{e_1^d})_{d-1}$ satisfy the first regularity condition, i. e. the chosen shelling of $\mathrm{bd}(e_1^d)$ is a regular order. If $k_d = 1$, we are done by now.

If $k_d > 1$, let $1 \leq \ell < k_d$. We assume: the basis elements in $\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}$ which may be ordered in a shelling satisfy the first regularity condition (i. e. the subcomplex generated by $\bigcup_{i=1}^{\ell} (\Omega_{e_i^d} \setminus \{e_i^d\})$ is regular). We want to show that this also holds for $\big(\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}\big) \cup (\Omega_{e_{\ell+1}^d})_{d-1}$.

If $e_{\ell+1}^d$ is precritical, then $\big(\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}\big) \cup (\Omega_{e_{\ell+1}^d})_{d-1} = \bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}$, and there is nothing left to do. So we assume that $e_{\ell+1}^d$ is not precritical. Let

$$\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1} = \{e_1, \ldots, e_r, f_1, \ldots, f_s\}$$
$$\text{and} \quad (\Omega_{e_{\ell+1}^d})_{d-1} = \{f_1, \ldots, f_s, h_1, \ldots, h_t\}.$$

We have $t \geq 1$ and, because of shellability, $s \geq 1$. By assumption, the first regularity condition is fulfilled by the elements of $\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}$, so we have only to consider $h_1, \ldots, h_t$.

Let there be some $1 \leq k \leq t$ so that

$$\mathrm{bd}(h_k) \subset \Big(\bigcup_{i=1}^{r} \mathrm{bd}(e_i)\Big) \cup \Big(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\Big) \cup \underbrace{\Big(\bigcup_{i=1}^{k-1} \mathrm{bd}(h_i)\Big)}_{=\emptyset \text{ if } k=1}.$$

As the chain complex C is shellable we know that

$$\Big(\bigcup_{i=1}^{\ell} \Omega_{e_i^d}\Big) \cap \Omega_{e_{\ell+1}^d} = \bigcup_{i=1}^{s} \Omega_{f_i}.$$

Therefore, $\mathrm{bd}(h_k) \cap \big(\bigcup_{i=1}^{r} \mathrm{bd}(e_i)\big) \subset \big(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\big)$, hence:

$$\mathrm{bd}(h_k) \subset \Big(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\Big) \cup \Big(\bigcup_{i=1}^{k-1} \mathrm{bd}(h_i)\Big).$$

As the second regularity condition holds for the elements in $(\Omega_{e_{\ell+1}^d})_{d-1}$ there exist elements $a_1, \ldots, a_s, b_1, \ldots, b_k \in R$ with $b_k \neq 0$ so that

$$\begin{aligned} b_k \partial_{d-1}(h_k) &= \sum_{i=1}^{s} a_i \partial_{d-1}(f_i) + \sum_{j=1}^{k-1} b_j \partial_{d-1}(h_j) \\ &= \sum_{i=1}^{r} 0 \cdot \partial_{d-1}(e_i) + \sum_{i=1}^{s} a_i \partial_{d-1}(f_i) + \sum_{j=1}^{k-1} b_j \partial_{d-1}(h_j) \end{aligned}$$

Hence, the first regularity condition is fulfilled, so $\mathrm{sk}_{d-1}(C)$ is a regular chain complex.

Additionally, every subcomplex $C_{e_i^\nu}$ of $\mathrm{sk}_{d-1}(C)$ is acyclic if this holds for C. So, if C is totally regular, then $\mathrm{sk}_{d-1}(C)$ is totally regular, too. □

Theorem 6.7. *Let (C, Ω) be a regular chain complex, finite of order $d \geq 1$. Then the $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C is also regular. If (C, Ω) is even totally regular, $\mathrm{sk}_{d-1}(C)$ is also totally regular.*

Proof. Lemma 6.6 deals with this statement for pure regular chain complexes so there is only the non-pure case to do.

It is clear that the $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C satisfies the second regularity condition so we have to prove only the first one.

Let $\Gamma := \{e \in \Omega \mid e \notin \mathrm{bd}(f) \text{ for all } f \in \Omega\}$ be the subset of Ω of all maximal basis elements. Let the elements of Γ be ordered in a regular order.

Let $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\} \subset \Gamma$, and $\Gamma \cap \Omega_{d-1} = \{g_1^{d-1}, \ldots, g_m^{d-1}\}$. As a regular order is always monotonically descending, in the chosen regular order of Γ the elements of Ω_d come first, followed by all elements of $\Gamma \cap \Omega_{d-1}$. The basis of the chain module $\big(\mathrm{sk}_{d-1}(C)\big)_{d-1}$ is

$$\Omega_{d-1} = \Big(\bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)\Big) \cup \{g_1^{d-1}, \ldots, g_m^{d-1}\}.$$

The subcomplex $\widehat{C}$ of C with basis $\bigcup_{i=1}^{k_d} \Omega_{e_i^d}$ is pure and regular. The basis of its chain module $\widehat{C}_{d-1}$ is $\widehat{\Omega}_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$.

By Lemma 6.6, the elements in $\widehat{\Omega}_{d-1}$ satisfy the first regularity condition. The same holds for $\{g_1^{d-1}, \ldots, g_m^{d-1}\}$ because C is regular. Hence, the $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C)$ of C is regular.

If C is totally regular, then any subcomplex $C_{e_i^\nu}$ of $\mathrm{sk}_{d-1}(C) \subset C$ is acyclic, so $\mathrm{sk}_{d-1}(C)$ is totally regular. □

The next two corollaries follow directly from Theorem 6.7.

Corollary 6.8. *Let (C, Ω) be a regular chain complex, finite of order d. For $0 \leq i \leq d$, every i-skeleton $\mathrm{sk}_i(C)$ of C is regular.*

Corollary 6.9. *Let (C, Ω) be a totally regular chain complex, finite of order d. For $0 \leq i \leq d$, every i-skeleton $\mathrm{sk}_i(C)$ of C is totally regular.*

6.3. **Homology of Pure Totally Regular Chain Complexes.** We give a description of the homology of totally regular chain complexes. We will start with pure complexes and compute the homology for a special case. But first we need some facts about reduced homology of totally regular chain complexes.

Lemma 6.10. *Let (C, Ω) be a totally regular chain complex of order d. There is no element $x \in C_1$ so that ${}^\#\mathrm{bd}(x) = 1$.*

Proof. Let $\Omega_1 := \{e_1^1, \ldots, e_{k_1}^1\}$ and $\Omega_0 := \{e_1^0, \ldots, e_{k_0}^0\}$ be the bases of the chain modules C_1 resp. C_0. Because C is totally regular, these orderings of Ω_1 and Ω_0 are shellings and every subcomplex $C_{e_i^1}$ is acyclic for $1 \leq i \leq k_1$. Then we know by Lemma 4.5 that ${}^\#\mathrm{bd}(e_i^1) = 2$ for $1 \leq i \leq k_1$.

We assume that an element $x \in C_1$ exists so that ${}^\#\mathrm{bd}(x) = 1$. Let $i_0 := \max\{1 \leq i \leq k_1 \mid a_i \neq 0\}$, so we get:

$$x = \sum_{i=1}^{i_0-1} a_i e_i^1 + a_{i_0} e_{i_0}^1.$$

As the 1-skeleton of C is shellable $\mathrm{bd}(e_{i_0}^1) \cap \left(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\right) \neq \emptyset$. So we distinguish two cases:

- $\mathrm{bd}(e_{i_0}^1) \subset \left(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\right)$. Because the 1-skeleton of C is also totally regular due to Lemma 6.9 there are λ_i for $1 \leq i \leq i_0$, $\lambda_{i_0} \neq 0$, so that
$$\lambda_{i_0} \partial_1(e_{i_0}^1) = \sum_{i=1}^{i_0-1} \lambda_i \partial_1(e_i^1).$$
Therefore, we get:
$$\begin{aligned} \partial_1(\lambda_{i_0} x) &= \sum_{i=1}^{i_0-1} a_i \lambda_{i_0} \partial_1(e_i^1) + a_{i_0} \lambda_{i_0} \partial_1(e_{i_0}^1) \\ &= \sum_{i=1}^{i_0-1} \left(a_i \lambda_{i_0} + a_{i_0} \lambda_i\right) \partial_1(e_i^1), \end{aligned}$$
and ${}^\#\mathrm{bd}(\lambda_{i_0} x) = 1$ because of $\partial_1(\lambda_{i_0} x) = \lambda_{i_0} \partial_1(x)$.
- ${}^\#\left(\mathrm{bd}(e_{i_0}^1) \cap \left(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\right)\right) = 1$. Then $\mathrm{bd}(x) = \mathrm{bd}(e_{i_0}^1) \setminus \mathrm{bd}\left(\sum_{i=1}^{i_0-1} a_i e_i^1\right)$. So we get ${}^\#\mathrm{bd}\left(\sum_{i=1}^{i_0-1} a_i e_i^1\right) = 1$.

In both cases, we get an element of C_1 which is a linear combination of $e_1^1, \ldots, e_{i_0-1}^1$ having only one element in its boundary. Iterating this way delivers a contradiction as ${}^\#\mathrm{bd}(e_1^1) = 2$. □

By Lemma 3.15 there is a R-linear mapping $\epsilon\colon C_0 \to R$ with $\epsilon \circ \partial_1 = 0$ and $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0$ for any totally regular chain complex (C, Ω). Hence, for

totally regular chain complexes reduced and nonreduced homology are different: $H_0(C) \cong \widetilde{H}_0(C) \oplus R$. We need this fact in the proof of the following theorem.

Theorem 6.11. *Let (C, Ω) be a pure totally regular chain complex of order d. Let there be only noncritical basis elements in Ω_d. Then, the chain complex (C, Ω) is acyclic.*

Remark 6.12. The chain complex (C, Ω) above is pure, totally regular without precritical basis elements. If $\Omega_d = \{e_1^d, \dots, e_{k_d}^d\}$ is ordered in a regular order, the first regularity condition delivers the following for such a chain complex: For any $e_\ell^d \in \Omega_d$, $2 \le \ell \le k_d$, there exists an element $e_{j_\ell}^{d-1} \in \mathrm{bd}(e_\ell^d)$ so that

$$e_{j_\ell}^{d-1} \notin \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d).$$

Proof of Theorem 6.11. We use induction to the order d.

If $d = 0$, then (C, Ω) is a chain complex of order 0 without precritical basis elements in Ω_0. Hence, $\Omega_0 = \{e_1^0\}$ and $C_0 = \langle e_1^0 \rangle$. Therefore, $H_0(C) \cong R$, i. e. the chain complex C is acyclic.

Let $d \ge 1$. We assume that the theorem's statement is true for pure totally regular chain complexes of order $(d-1)$ without precritical basis elements.

Let $\Omega_d := \{e_1^d, \dots, e_{k_d}^d\}$, so that its basis elements are ordered in a regular order. For $1 \le \ell \le k_d$ we define:

$$\Phi_\ell := \bigcup_{i=1}^{\ell} \Omega_{e_i^d}.$$

We get $\Phi_{\ell-1} \cap \Omega_{e_\ell^d} = \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}$ for $2 \le \ell \le k_d$. About the chain complex $P_\ell \subset C$ having $\Phi_{\ell-1} \cap \Omega_{e_\ell^d}$ as basis we know the following:

(1) P_ℓ is a pure chain complex of order $(d-1)$ and shellable because C is shellable.
(2) Every subcomplex $C_{e_i^\nu}$ of P_ℓ is shellable and acyclic.
(3) For every $e_i^\nu \in (\Phi_{\ell-1} \cap \Omega_{e_\ell^d})$ holds: $\Omega_{e_i^\nu} \subset (\Phi_{\ell-1} \cap \Omega_{e_\ell^d})$. Hence, the second regularity condition holds for P_ℓ.
(4) P_ℓ is a subcomplex of $C_{e_\ell^d}$. As e_ℓ^d fulfils the second regularity condition and $(\Phi_{\ell-1} \cap \Omega_{e_\ell^d})_{d-1} = \left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}\right)_{d-1} \subset (\Omega_{e_\ell^d})_{d-1}$ we conclude that P_ℓ satisfies the first regularity condition.

Hence P_ℓ is a pure totally regular chain complex of order $(d-1)$. If P_ℓ has no precritical elements in $(\Phi_{\ell-1} \cap \Omega_{e_\ell^d})_{d-1}$, we can apply our induction hypothesis, i. e. P_ℓ is acyclic then.

So we have to show that P_ℓ has no precritical elements in $(\Phi_{\ell-1} \cap \Omega_{e_\ell^d})_{d-1}$. We will do this separately for $d = 1$ and for $d \ge 2$.

$d = 1$**:** We consider the chain complex $C_1 \to C_0 \to 0$ with $C_1 = \langle e_1^1, \dots, e_{k_1}^1 \rangle$, $k_1 \ge 1$, about which we know the following:

(1) $\Omega_1 = \{e_1^1, \dots, e_{k_1}^1\}$ has no precriticical basis elements.
(2) Every subcomplex $C_{e_i^1}$ is acyclic by definition. Due to Lemma 4.5 we know that each chain module $(C_{e_i^1})_0$ is generated by exactly two elements, i. e. ${}^\#(\Omega_{e_i^1})_0 = 2$.
(3) $\Phi_{\ell-1} \cap \Omega_{e_\ell^1} = \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right) \cap \Omega_{e_\ell^1} \neq \emptyset$ for $2 \le \ell \le k_1$ as the chain complex C is shellable.

So we get $^{\#}(\Phi_{\ell-1} \cap \Omega_{e^1_\ell}) = {}^{\#}(\Phi_{\ell-1} \cap \Omega_{e^1_\ell})_0 \geq 1$. Furthermore, we know that

$$\Phi_{\ell-1} \cap \Omega_{e^1_\ell} = \left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e^1_i} \right) \cap \Omega_{e^1_\ell} \right)_0 \subset (\Omega_{e^1_\ell})_0.$$

As $^{\#}(\Omega_{e^1_i})_0 = 2$ we conclude: $^{\#}(\Phi_{\ell-1} \cap \Omega_{e^1_\ell})_0 \leq 2$. If $^{\#}(\Phi_{\ell-1} \cap \Omega_{e^1_\ell})_0 = 2$, then e^1_ℓ would be precritical which contradicts our assumption.

Therefore $^{\#}(\Phi_{\ell-1} \cap \Omega_{e^1_\ell})_0 = 1$ for $2 \leq \ell \leq k_1$, so $\Phi_{\ell-1} \cap \Omega_{e^1_\ell}$ has no precritical basis elements.

$d \geq 2$: For $2 \leq \ell \leq k_d$ let $(\Phi_{\ell-1} \cap \Omega_{e^d_\ell})_{d-1} = \{g_1, \ldots, g_{m_\ell}\}$, ordered in a regular order. If $m_\ell = 1$, there is nothing left to do, so let $m_\ell \geq 2$.

We assume: There is some $2 \leq j \leq m_\ell$ so that g_j is precritical:

$$a_j \partial_{d-1} g_j = \sum_{i=1}^{j-1} a_i \partial_{d-1} g_i \quad \text{with } a_i \in R \text{ for } 1 \leq i \leq j \text{ and } a_j \neq 0.$$

So there holds: $a_j g_j - \sum\limits_{i=1}^{j-1} a_i g_i \in \ker \partial_{d-1}$.

By assumption, the chain complex C has no precritical elements in Ω_d. Therefore, $(\Phi_{\ell-1} \cap \Omega_{e^d_\ell})_{d-1} \subsetneq (\Omega_{e^d_\ell})_{d-1}$. Otherwise, we would get:

$$\begin{aligned}(\Omega_{e^d_\ell})_{d-1} = \mathrm{bd}(e^d_\ell) \subset (\Phi_{\ell-1})_{d-1} &= \left(\bigcup_{i=1}^{\ell-1} \Omega_{e^d_i} \right)_{d-1} \\ &= \bigcup_{i=1}^{\ell-1} (\Omega_{e^d_i})_{d-1} = \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e^d_i),\end{aligned}$$

and because of the first regularity condition e^d_ℓ would be precritical then. ↯

Hence, $\partial_d(e^d_\ell) = \sum\limits_{i=1}^{m_\ell} b_i g_i + r$ with $\sum\limits_{i=1}^{m_\ell} b_i g_i \in \langle \Phi_{\ell-1} \cap \Omega_{e^d_\ell} \rangle$ and some $r \in \langle (\Omega_{e^d_\ell})_{d-1} \setminus \Phi_{\ell-1} \rangle$. As $(\Omega_{e^d_\ell})_{d-1} = \mathrm{bd}(e^d_\ell)$ it is $r \neq 0$.

Furthermore, we know $\partial_{d-1} \circ \partial_d(e^d_\ell) = 0$, i. e. $\partial_d(e^d_\ell) \in \ker \partial_{d-1}$. So, as $r \neq 0$, there are two linearly independent elements in $\ker \partial_{d-1}$, namely $\partial_d(e^d_\ell)$ and $a_j g_j - \sum_{i=1}^{j-1} a_i g_i$. These both elements are even contained in $(C_{e^d_\ell})_{d-1}$, so we conclude: $\ker \partial_{d-1}|_{C_{e^d_\ell}}$ is generated by at least two elements.

We know that $(C_{e^d_\ell})_d = \langle e^d_\ell \rangle$, therefore $\mathrm{im}\, \partial_d|_{C_{e^d_\ell}}$ is generated by one element. Hence we conclude: $H_{d-1}(C_{e^d_\ell}) \neq 0$ ↯– a contradiction to our assumption that $C_{e^d_\ell}$ is acyclic, i. e. $H_{d-1}(C_{e^d_\ell}) = 0$ if $d \geq 2$.

Therefore, $(\Phi_{\ell-1} \cap \Omega_{e^d_\ell})_{d-1}$ contains no precritical basis elements.

By induction hypothesis we conclude for $2 \leq \ell \leq k_d$: The chain complex P_ℓ with basis $(\Phi_{\ell-1} \cap \Omega_{e^d_\ell})$ is acyclic.

Let Q_j be the chain complex with basis $\Phi_j = \bigcup_{i=1}^{j} \Omega_{e^d_i}$ for $1 \leq j \leq k_d$. Because Ω_d is ordered in a regular order, Q_j is a totally regular chain complex. In particular, $Q_1 = \langle \Omega_{e^d_1} \rangle = C_{e^d_1}$ and $Q_{k_d} = \langle \bigcup_{i=1}^{k_d} \Omega_{e^d_i} \rangle = C$. By assumption, Q_1 is an acyclic chain complex.

As $(Q_\ell)_n = \langle (\Phi_{\ell-1})_n \cup (\Omega_{e^d_\ell})_n \rangle = \langle (\Phi_{\ell-1})_n \rangle + \langle (\Omega_{e^d_\ell})_n \rangle = (Q_{\ell-1})_n + (C_{e^d_\ell})_n$ and $(P_\ell)_n = (Q_{\ell-1})_n \cap (C_{e^d_\ell})_n$ for any $2 \le \ell \le k_d$ and for any $n \ge 0$ we get the following exakt sequence:

$$0 \to (P_\ell)_n \xrightarrow{\varphi_n} (Q_{\ell-1})_n \oplus (C_{e^d_\ell})_n \xrightarrow{\psi_n} (Q_\ell)_n \to 0$$

with $\varphi_n(x) = (x, -x)$, $\psi_n(x, y) = x + y$ (cf. Hatcher, 2008, page 149). Because (C, Ω) is a totally regular chain complex there is some mapping $\epsilon \colon C_0 \to R$ with $\epsilon(e^0_i) \neq 0$ for all $e^0_i \in \Omega_0$. Hence each chain complex P_ℓ, $Q_{\ell-1}$, $C_{e^d_\ell}$ and Q_ℓ can be augmented by the restriction of ϵ (cf. Hatcher, 2008, page 150).

Furthermore, $\varphi_n \circ \partial_{n+1} = \partial_n \circ \varphi_{n+1}$ and $\psi_n \circ \partial_{n+1} = \partial_n \circ \psi_{n+1}$. As in Hatcher (2008, page 116) we obtain a long exact sequence of reduced homology groups:

$$\begin{aligned} \ldots \to \widetilde{H}_m(P_\ell) \xrightarrow{\varphi_{*n}} \widetilde{H}_m(Q_{\ell-1}) \oplus \widetilde{H}_m(C_{e^d_\ell}) \xrightarrow{\psi_{*n}} \widetilde{H}_m(Q_\ell) \xrightarrow{\delta_n} \widetilde{H}_{m-1}(P_\ell) \xrightarrow{\varphi_{*n-1}} \ldots \\ \ldots \xrightarrow{\delta_1} \widetilde{H}_0(P_\ell) \xrightarrow{\varphi_{*0}} \widetilde{H}_0(Q_{\ell-1}) \oplus \widetilde{H}_0(C_{e^d_\ell}) \xrightarrow{\psi_{*0}} \widetilde{H}_0(Q_\ell) \xrightarrow{\delta_0} 0. \end{aligned}$$

For an acyclic chain complex all reduced homology groups are 0. As P_ℓ, $Q_{\ell-1}$ and $C_{e^d_\ell}$ are all acyclic chain complexes we conclude that Q_ℓ is acyclic, too.

Therefore, $C = Q_{k_d}$ is an acyclic chain complex, i. e. $H_i(C) = 0$ for $i \ge 1$ and $H_0(C) \cong R$. □

Theorem 6.13. *Let (C, Ω) be a pure totally regular chain complex of order $d \ge 1$. Let the basis $\Omega_d = \{e^d_1, \ldots, e^d_{k_d}\}$ of C_d have $n < k_d$ precritical elements. Then there holds for the homology of C:*

$$\begin{aligned} &H_d(C) \cong R^n; \\ &H_i(C) = 0 \quad \text{for } i \neq 0, d; \\ &H_0(C) \cong R. \end{aligned}$$

Remark 6.14. $H_0(C) \cong R^{k_0}$ if $d = 0$, cf. Remark 3.12.

Proof. We can assume that all noncritical elements in Ω_d come first in the regular order. Otherwise we can change the order so that the precritical (and critical) elements of Ω_d come at last; this has no influence to the shellability and regularity of C. Let $m := k_d - n$, then we have:

$$\Omega_d = \{\underbrace{e^d_1, \ldots, e^d_m}_{\text{noncritical}}, \underbrace{e^d_{m+1}, \ldots, e^d_{k_d}}_{\text{precritical}}\}.$$

Due to Theorem 3.14 we know that $H_d(C) \cong R^n$.

Consider now the chain complex $\widehat{C} := \langle \bigcup_{i=1}^m \Omega_{e^d_i} \rangle$ which is a pure subcomplex of C of order d. Its chain modules are $\widehat{C}_d = \langle e^d_1, \ldots, e^d_m \rangle$ and $\widehat{C}_\nu = C_\nu$ for $0 \le \nu \le d-1$. As C is a totally regular chain complex this holds for $\widehat{C}$, too. In opposite to C the chain complex $\widehat{C}$ has no precritical elements. Due to Theorem 6.11 the complex $\widehat{C}$ is acyclic, i. e. $H_0(\widehat{C}) \cong R$ and $H_i(\widehat{C}) = 0$ for $i \ge 1$.

Because $\widehat{C}_{d-1} = C_{d-1}$ and $H_{d-1}(\widehat{C}) = 0$ we get

$$\operatorname{im} \partial_d|_{\widehat{C}_d} \subset \operatorname{im} \partial_d \subset \ker \partial_{d-1} = \ker \partial_{d-1}|_{\widehat{C}_{d-1}} = \operatorname{im} \partial_d|_{\widehat{C}_d}.$$

Therefore, $\operatorname{im}\partial_d|_{\widehat{C}_d} = \operatorname{im}\partial_d$. As $\widehat{C}_\nu = C_\nu$ for $0 \le \nu \le d-1$ we conclude:

$$\begin{aligned} H_i(C) &= H_i(\widehat{C}) = 0 \quad \text{for } 1 \le i \le d-1, \\ H_0(C) &= H_0(\widehat{C}) \cong R. \end{aligned}$$

□

6.4. **Homology of arbitrary Totally Regular Chain Complexes.** We consider the general case and compute the homology of arbitrary totally regular chain complexes.

Theorem 6.15. *Let (C, Ω) be a totally regular chain complex of order $d \ge 1$. For any $0 \le \nu \le d$ let $\Omega_\nu := \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$ be the basis of the chain module C_ν. Let Γ be the subset of Ω which contains all maximal basis elements. For $0 \le \nu \le d$, let there be $n_\nu < k_\nu$ precritical elements in $(\Gamma \cap \Omega_\nu)$. Then the homology of C is:*

$$\begin{aligned} H_i(C) &\cong R^{n_i} \quad \textit{for } 1 \le i \le d; \\ H_0(C) &\cong R^{n_0+1}. \end{aligned}$$

Proof. We assume that Γ is ordered in a regular order. Then each Ω_ν is ordered so that the elements of Γ come at last; let

$$\Omega_\nu := \{\underbrace{e_1^\nu, \dots, e_{m_\nu}^\nu}_{\notin \Gamma}, \underbrace{e_{m_\nu+1}^\nu, \dots, e_{k_\nu}^\nu}_{\in \Gamma}\}.$$

So, we even have $n_\nu \le k_\nu - m_\nu$.

We consider the chain complex $C^d := \langle \bigcup_{i=1}^{k_d} \Omega_{e_i^d} \rangle$ which is a subcomplex of C. It is pure of order d and totally regular. The reader may notice that $C^d = C$ if $\Gamma = \Omega_d$. According to Theorem 6.13 we know:

$$\begin{aligned} H_d(C^d) &\cong R^{n_d}; \\ H_i(C^d) &= 0 \quad \text{for } i \ne 0, d; \\ H_0(C^d) &\cong R. \end{aligned}$$

As $(C^d)_d = C_d$ we get $H_d(C) \cong R^{n_d}$.

For $1 \le \mu \le d-1$ we consider the chain complex $C^\mu := \langle \bigcup_{i=1}^{k_\mu} \Omega_{e_i^\mu} \rangle \subset \operatorname{sk}_\mu(C)$ which is pure and finite of order μ. We know that each μ-skeleton of C is shellable and totally regular. Let $\Omega_{\operatorname{sk}_\mu}$ be the basis of $\operatorname{sk}_\mu(C)$ and $\Gamma_{\operatorname{sk}_\mu} := \{e \in \Omega_{\operatorname{sk}_\mu} \mid e \notin \operatorname{bd}(f) \text{ for all } f \in \Omega_{\operatorname{sk}_\mu}\}$, ordered in a regular order. As a regular order is always monotonically descending, the elements $e_1^\mu, \dots, e_{k_\mu}^\mu$ come first in the regular order of $\Gamma_{\operatorname{sk}_\mu}$. Therefore, the complex C^μ is also shellable and totally regular.

Let there be ℓ_μ precritical basis elements in $\{e_1^\mu, \dots, e_{m_\mu}^\mu\}$. By Theorem 6.13 we get:

$$\begin{aligned} H_\mu(C^\mu) &\cong R^{n_\mu + \ell_\mu}; \\ H_i(C^\mu) &= 0 \quad \text{for } i \ne 0, \mu; \\ H_0(C^\mu) &\cong R. \end{aligned}$$

Therefore, we also know $H_\mu(C^{\mu+1}) = 0$. Because $(C^{\mu+1})_{\mu+1} = C_{\mu+1}$ we get:

$$\operatorname{im}\partial_{\mu+1} = \operatorname{im}\partial_{\mu+1}|_{C^{\mu+1}} = \ker\partial_\mu|_{\langle e_1^\mu, \dots, e_{m_\mu}^\mu \rangle}.$$

As the chain complex $\widehat{C}^\mu := \langle \bigcup_{i=1}^{m_\mu} \Omega_{e_i^\mu} \rangle \subset C^\mu$ is pure of order μ and totally regular, there holds due to Theorem 6.13: $H_\mu(\widehat{C}^\mu) \cong \ker\partial_\mu|_{\langle e_1^\mu, \dots, e_{m_\mu}^\mu \rangle} \cong R^{\ell_\mu}$.

It is $\Gamma \cap \Omega_\mu = \{e^\mu_{m_\mu+1}, \dots, e^\mu_{k_\mu}\}$, hence $e^\mu_i \notin \operatorname{im} \partial_{\mu+1}$ for $m_\mu + 1 \leq i \leq k_\mu$. We conclude:

$$H_\mu(C) = \ker \partial_\mu / \operatorname{im} \partial_{\mu+1} \cong R^{n_\mu} \quad \text{for } 1 \leq \mu \leq d-1.$$

For $\mu = 0$ we consider the chain complex $C^1 := \langle \bigcup_{i=1}^{k_1} \Omega_{e^1_i} \rangle \subset \mathrm{sk}_1(C)$ which is pure of order 1 and totally regular. Hence, $H_0(C^1) = \langle e^0_1, \dots, e^0_{m_0} \rangle / \operatorname{im} \partial_1 \cong R$. Because $e^0_i \notin \operatorname{im} \partial_1$ for $m_0 + 1 \leq i \leq k_0$ we get:

$$H_0(C) = C_0 / \operatorname{im} \partial_1 \cong R^{n_0+1}. \qquad \square$$

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